Exercise 1

Show that the point $z = 0$ is a simple pole of the function

$$f(z) = \csc z = \frac{1}{\sin z}$$

and that the residue there is unity by appealing to

(a) Theorem 2 in Sec. 76;

(b) the Laurent series for $\csc z$ that was found in Exercise 2, Sec. 67.

Solution

Let $q(z) = \sin z$ so that

$$f(z) = \frac{1}{q(z)}. \tag{1}$$

Since $q(0) = 0$ and $q'(0) \neq 0$, $q(z)$ has a zero of order 1 at $z = 0$ and can be written as

$$q(z) = zg(z),$$

where $g(0) \neq 0$. So then

$$\begin{aligned} f(z) &= \frac{1}{zg(z)} \\ &= \frac{1/g(z)}{z}. \end{aligned}$$

From this form we see that $z = 0$ is a pole of order 1 (a simple pole) of $f(z)$.

Part (a)

Using Theorem 2 in Sec. 76, the residue of $f(z)$ at $z = 0$ is calculated by

$$\operatorname*{Res}_{z=0} f(z) = \frac{p(0)}{q'(0)},$$

where $p(z)$ is set to be the function in the numerator and $q(z)$ is set to be the function in the denominator.

$$\begin{array}{lclcl} p(z) = 1 & & & \Rightarrow & p(0) = 1 \\ q(z) = \sin z & \rightarrow & q'(z) = \cos z & \Rightarrow & q'(0) = 1 \end{array}$$

Therefore,

$$\operatorname*{Res}_{z=0} f(z) = 1.$$

Part (b)

The residue can also be found by writing out the series expansion of $\sin z$ about $z = 0$, performing the long division, and noting the coefficient of $1/z$. We reach the same conclusion.

$$f(z) = \frac{1}{\sin z} = \frac{1}{z - \frac{z^3}{6} + \cdots} = \frac{1}{z} + \frac{z}{6} + \cdots$$